

# RESONANCES AND BALLS IN OBSTACLE SCATTERING WITH NEUMANN BOUNDARY CONDITIONS

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ABSTRACT. We consider scattering by an obstacle in  $\mathbb{R}^d$ ,  $d \geq 3$  odd. We show that for the Neumann Laplacian if an obstacle has the same resonances as the ball of radius  $\rho$  does, then the obstacle is a ball of radius  $\rho$ . We give related results for obstacles which are disjoint unions of several balls of the same radius.

## 1. INTRODUCTION

The purpose of this note is to show that for obstacle scattering in  $\mathbb{R}^d$ ,  $d \geq 3$  odd, a ball is uniquely determined by its resonances for the Laplacian with Neumann boundary conditions. We actually show a somewhat stronger result: If  $\mathcal{O}_1$  and  $\mathcal{O}_2$  have the same (Neumann) resonances and  $\mathcal{O}_1$  is the disjoint union of  $m$  balls, each of radius  $\rho$ , then so is  $\mathcal{O}_2$ .

We fix some notation. Let  $\mathcal{O} \subset \mathbb{R}^d$  be a compact set of dimension  $d$  with smooth boundary  $\partial\mathcal{O}$ . Let  $\Delta_{\mathbb{R}^d \setminus \mathcal{O}}$  denote the Laplacian with *Neumann* boundary conditions on  $\mathbb{R}^d \setminus \mathcal{O}$ . Set  $R_{\mathcal{O}}(\lambda) = (\Delta_{\mathbb{R}^d \setminus \mathcal{O}} - \lambda^2)^{-1}$ , with the convention that  $R_{\mathcal{O}}(\lambda)$  is bounded on  $L^2(\mathbb{R}^d \setminus \mathcal{O})$  when  $\Im \lambda > 0$ . Then, for odd  $d$ , it is known that  $R_{\mathcal{O}}(\lambda) : L^2_{\text{comp}}(\mathbb{R}^d \setminus \mathcal{O}) \rightarrow L^2_{\text{loc}}(\mathbb{R}^d \setminus \mathcal{O})$  has a meromorphic continuation to  $\mathbb{C}$ . Set

$$\mathcal{R}_{\mathcal{O}} = \{\lambda_0 \in \mathbb{C} : R_{\mathcal{O}}(\lambda) \text{ has a pole at } \lambda_0, \text{ repeated according to multiplicity}\}.$$

The multiplicity can be defined via

$$m_{\mathcal{O}}(\lambda) = \text{rank} \oint_{|z-\lambda|=\epsilon} R_{\mathcal{O}}(z) dz, \quad 0 < \epsilon \ll 1.$$

Set  $B(\rho)$  to be the closed ball of radius  $\rho$  centered at the origin.

**Theorem 1.1.** *If  $d \geq 3$  is odd, and  $\mathcal{O} \subset \mathbb{R}^d$  is a smooth compact set with  $\mathcal{R}_{\mathcal{O}} = \mathcal{R}_{B(\rho)}$ , then  $\mathcal{O}$  is a translate of  $B(\rho)$ .*

We note that a simple scaling argument shows that if  $\rho_1 \neq \rho_2$  then  $\mathcal{R}_{B(\rho_1)} \neq \mathcal{R}_{B(\rho_2)}$  so that  $B(\rho)$  is determined (up to translation) by its Neumann resonances.

Hassell and Zworski [7] proved that for the *Dirichlet* Laplacian in  $\mathbb{R}^3$ , if a connected set  $\mathcal{O}$  has the same resonances as  $B(\rho)$ , then  $\mathcal{O}$  is a translate of  $B(\rho)$ . It seems that their argument also works for Neumann boundary conditions, again in dimension 3 with the restriction that  $\mathcal{O}$  is connected. That we can prove this result for all odd dimensions for Neumann boundary conditions follows partly from the fact that for the Neumann case the resonances determine the

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determinant of the scattering matrix up to one unknown constant, while for the Dirichlet case there are two unknown constants— compare Lemma 2.1 to [7].

We shall actually prove the following theorem, from which Theorem 1.1 follows immediately.

**Theorem 1.2.** *Let  $d \geq 3$  be odd. Let  $\mathcal{O}_1 \subset \mathbb{R}^d$  be a disjoint union of  $m$  closed balls each of the same radius  $\rho$ , and let  $\mathcal{O}_2 \subset \mathbb{R}^d$  be a smooth compact set. If  $\mathcal{R}_{\mathcal{O}_1} = \mathcal{R}_{\mathcal{O}_2}$ , then  $\mathcal{O}_2$  is also a disjoint union of  $m$  closed balls of radius  $\rho$ .*

The proof of Theorem 1.2 uses heat coefficients. These are closely related to the singularity at  $t = 0$  of the distribution

$$(1) \quad u_{\mathcal{O}}(t) = \text{Tr} \left( \cos \left( t \sqrt{\Delta_{\mathbb{R}^d \setminus \mathcal{O}}} \right) - \cos \left( t \sqrt{\Delta_{\mathbb{R}^d}} \right) \right).$$

This is rather informal. One way to make precise sense of (1) is as follows. With  $\rho'$  chosen sufficiently large that  $\mathcal{O} \subset B(\rho')$ ,

$$(2) \quad u_{\mathcal{O}}(t) = \text{Tr} \left( \cos \left( t \sqrt{\Delta_{\mathbb{R}^d \setminus \mathcal{O}}} \right) - \mathbb{1}_{\mathbb{R}^d \setminus B(\rho')} \cos \left( t \sqrt{\Delta_{\mathbb{R}^d}} \right) \mathbb{1}_{\mathbb{R}^d \setminus B(\rho')} \right) + \text{Tr} \left( \mathbb{1}_{B(\rho')} \cos \left( t \sqrt{\Delta_{\mathbb{R}^d}} \right) \mathbb{1}_{B(\rho')} \right)$$

where  $\mathbb{1}_E$  is the characteristic function of the set  $E$ , compare [15]. The Poisson formula for resonances in odd dimensions is

$$u_{\mathcal{O}}(t) = \sum_{\lambda_j \in \mathcal{R}_{\mathcal{O}}} e^{i\lambda_j |t|}, \quad t \neq 0$$

see [3, 8, 9, 14], and [13] for an application to the existence of resonances. This shows that any singularities of the distribution  $u_{\mathcal{O}}(t)$  at nonzero time are determined by the resonances of  $\Delta_{\mathbb{R}^d \setminus \mathcal{O}}$ . When  $\mathcal{O}$  is not connected, we expect that the distribution of (2) has singularities at nonzero times, see [10]. In particular, if  $\mathcal{O}$  consists of two disjoint convex obstacles, the distribution (2) has a singularity at twice the distance between them (e.g. [6] or [10, Section 6.4]). From this and Theorem 1.2 we have

**Corollary 1.3.** *Let  $d \geq 3$  be odd, and let  $\mathcal{O}_1, \mathcal{O}_2 \subset \mathbb{R}^d$  be smooth compact sets. Suppose  $\mathcal{O}_1$  is the disjoint union of 2 balls of radius  $\rho$  a distance  $\delta > 0$  apart. If  $\mathcal{R}_{\mathcal{O}_1} = \mathcal{R}_{\mathcal{O}_2}$ , then  $\mathcal{O}_2$  is obtained from  $\mathcal{O}_1$  by a rigid motion.*

This paper was inspired by [7], and we use some of the same notation.

## 2. PROOF OF THEOREM 1.2

Let  $S_{\mathcal{O}}(\lambda)$  be the scattering matrix for the Neumann Laplacian on  $\mathbb{R}^d \setminus \mathcal{O}$ , and set  $s_{\mathcal{O}}(\lambda) = \det S_{\mathcal{O}}(\lambda)$ . Set

$$E(z) = (1 - z) \exp \left( \sum_1^d \frac{z^j}{j} \right).$$

In the lemma below we have not made any assumptions on the connectivity of  $\mathbb{R}^d \setminus \mathcal{O}$ . If  $\mathbb{R}^d \setminus \mathcal{O}$  has a bounded component, then  $\mathcal{R}_{\mathcal{O}}$  contains the square roots, and the negative square roots, of eigenvalues of the Neumann Laplacian on the bounded component(s). These do not cause poles of the determinant of the scattering matrix. We remark that it is possible to have  $0 \in \mathcal{R}_{\mathcal{O}}$  here, if  $\mathbb{R}^d \setminus \mathcal{O}$  has a bounded component, but not if  $\mathbb{R}^d \setminus \mathcal{O}$  is connected.

**Lemma 2.1.** *Let  $d \geq 3$  be odd and  $\mathcal{O} \subset \mathbb{R}^d$  be a smooth compact set. Then*

$$s_{\mathcal{O}}(\lambda) = e^{ic_{\mathcal{O}}\lambda^d} \prod_{\lambda_j \in \mathcal{R}_{\mathcal{O}}, \lambda_j \neq 0} \frac{E(-\lambda/\lambda_j)}{E(\lambda/\lambda_j)}$$

for some real constant  $c_{\mathcal{O}}$ .

For a large number of more general self-adjoint operators (for example, for the Laplacian with different boundary conditions), the resonances determine the determinant of the scattering matrix up to two unknown constants, e.g. [7].

*Proof.* It follows from [14] that

$$s_{\mathcal{O}}(\lambda) = e^{ig_{\mathcal{O}}(\lambda)} \prod_{\lambda_j \in \mathcal{R}_{\mathcal{O}}, \lambda_j \neq 0} \frac{E(-\lambda/\lambda_j)}{E(\lambda/\lambda_j)}$$

where  $g_{\mathcal{O}}(\lambda)$  is a polynomial of degree at most  $d$ . It remains only to show that  $g_{\mathcal{O}}(\lambda) = c_{\mathcal{O}}\lambda^d$ .

We now use the expression of [11] for the scattering matrix. Choose  $\rho' > 0$  so that  $\mathcal{O}$  is contained in the ball of radius  $\rho'$  centered at the origin. For  $\psi \in C_c^{\infty}(\mathbb{R}^d)$ , let

$$\mathbb{E}_{\pm}^{\psi}(\lambda) : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{S}^{d-1})$$

be the operator with Schwartz kernel  $\psi(x) \exp(\pm i\lambda \langle x, \omega \rangle)$ . For  $i = 1, 2, 3$ , choose  $\chi_i \in C_c^{\infty}(\mathbb{R}^d)$  so that  $\chi_i \equiv 1$  if  $|x| < 4 + \rho'$  and  $\chi_{i+1} \equiv 1$  on the support of  $\chi_i$ ,  $i = 1, 2$ . Set

$$(3) \quad A_{\mathcal{O}}(\lambda) = \tilde{c}_d \lambda^{d-2} \mathbb{E}_+^{\chi_3}(\lambda) [\Delta_{\mathbb{R}^d}, \chi_1] R_{\mathcal{O}}(\lambda) [\Delta_{\mathbb{R}^d}, \chi_2]^t \mathbb{E}_-^{\chi_3}(\lambda).$$

Here  $\tilde{c}_d = i\pi(2\pi)^{-d}$  and  ${}^t \mathbb{E}_-^{\chi_3}$  denotes the transpose of  $\mathbb{E}_-^{\chi_3}$ . Then the scattering matrix  $S_{\mathcal{O}}(\lambda)$  associated to  $P$  is given by

$$(4) \quad S_{\mathcal{O}}(\lambda) = I + A_{\mathcal{O}}(\lambda).$$

Next, note that for the Neumann Laplacian,

$$R_{\mathcal{O}}(0) [\Delta_{\mathbb{R}^d}, \chi_2]^t \mathbb{E}_-^{\chi_3}(0) = \chi_2$$

since  $[\Delta_{\mathbb{R}^d}, \chi_2]^t \mathbb{E}_-^{\chi_3}(0) = \Delta_{\mathbb{R}^d} \chi_2$ , and  $\chi_2$  satisfies the Neumann boundary conditions. But then

$$\mathbb{E}_+^{\chi_3}(0) [\Delta_{\mathbb{R}^d}, \chi_1] \chi_2 = \int_{\mathbb{R}^d} \Delta_{\mathbb{R}^d} \chi_1 = 0$$

so that  $A_{\mathcal{O}}(\lambda)$  vanishes to order at least  $d-1$  at  $\lambda = 0$ . Thus

$$(5) \quad s_{\mathcal{O}}(\lambda) - 1 = \det(I + A_{\mathcal{O}}(\lambda)) - 1 = O(|\lambda|^{d-1})$$

at  $\lambda = 0$ . Using the fact that  $E(\lambda/\lambda^j) = O(|\lambda|^d)$  at  $\lambda = 0$ , we see that  $g^{(j)}(\lambda) = 0$  for  $j = 0, \dots, d-2$ . It is also well known that for  $\lambda \in \mathbb{R}$ ,  $s_{\mathcal{O}}(\lambda) s_{\mathcal{O}}(-\lambda) = 1$ , implying that  $g^{(d-1)}(0) = 0$ . Finally, since for  $\lambda \in \mathbb{R}$   $s_{\mathcal{O}}(\lambda) = s_{\mathcal{O}}(-\lambda)$ , we have that  $g^{(d)}(0)$  is real. This finishes the proof.  $\square$

We fix some notation. For a smooth compact set  $\mathcal{O} \subset \mathbb{R}^d$  with boundary  $\partial\mathcal{O}$  of dimension  $d-1$ , let  $\kappa_{1,\partial\mathcal{O}}, \dots, \kappa_{d-1,\partial\mathcal{O}}$  denote the principal curvatures of  $\partial\mathcal{O}$ . We use the normalization that the mean curvature  $H_{\partial\mathcal{O}}$  is  $H_{\partial\mathcal{O}} = \sum_1^{d-1} \kappa_{j,\partial\mathcal{O}}$ .

**Lemma 2.2.** *Let  $d \geq 3$  be odd and let  $\mathcal{O} \subset \mathbb{R}^d$  be a smooth compact set with boundary  $\partial\mathcal{O}$  of dimension  $d-1$ . Then  $\mathcal{R}_{\mathcal{O}}$  determines*

$$\text{Vol}(\partial\mathcal{O}), \int_{\partial\mathcal{O}} H_{\partial\mathcal{O}}, \text{ and } \int_{\partial\mathcal{O}} \left( 13H_{\partial\mathcal{O}}^2 + 2 \sum_1^{d-1} \kappa_{j,\partial\mathcal{O}}^2 \right).$$

*Proof.* As  $t \downarrow 0$ ,

$$(6) \quad \text{Tr}(e^{-t\Delta_{\mathbb{R}^d \setminus \mathcal{O}}} - e^{-t\Delta_{\mathbb{R}^d}}) \simeq t^{-d/2} \sum_{n=0}^{\infty} t^{n/2} a_n.$$

The trace in (6) can be made precise in exactly the same manner that (1) is made precise in (2). Explicit expressions for the first few  $a_n$  can be found in [4]. For Neumann boundary conditions in  $\mathbb{R}^d$ , we have

$$\begin{aligned} a_0 &= \alpha_0 \text{Vol}(\mathcal{O}) \\ a_1 &= \alpha_1 \text{Vol}(\partial\mathcal{O}) \\ a_2 &= \alpha_2 \int_{\partial\mathcal{O}} H_{\partial\mathcal{O}} \\ a_3 &= \alpha_3 \int_{\partial\mathcal{O}} \left( 13H_{\partial\mathcal{O}}^2 + 2 \sum_1^{d-1} \kappa_{j,\partial\mathcal{O}}^2 \right) \end{aligned}$$

where the  $\alpha_i$  are nonzero constants which depend on the dimension. Using Lemma 2.1, (6),

$$\text{Tr}(e^{-t\Delta_{\mathbb{R}^d \setminus \mathcal{O}}} - e^{-t\Delta_{\mathbb{R}^d}}) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-t\lambda^2} \frac{d}{d\lambda} \log s_{\mathcal{O}}(\lambda) d\lambda + \frac{1}{2} \sum_{\lambda_j \in \mathcal{R}_{\mathcal{O}} \cap \mathbb{R}} e^{-t\lambda_j^2}$$

(e.g. [12, Corollary 2.10] for the case of  $\mathbb{R}^d \setminus \mathcal{O}$  connected, or [5]) and the expressions for the  $a_i$  above, we prove the lemma.  $\square$

We are now ready for the proof of our theorem.

*Proof of Theorem 1.2.* Suppose  $\mathcal{O}_1, \mathcal{O}_2$  are as in the statement of the theorem, with  $\mathcal{R}_{\mathcal{O}_1} = \mathcal{R}_{\mathcal{O}_2}$ . Then by Lemma 2.2,

$$(7) \quad \begin{aligned} \text{Vol}(\partial\mathcal{O}_1) &= \text{Vol}(\partial\mathcal{O}_2), \quad \int_{\partial\mathcal{O}_1} H_{\partial\mathcal{O}_1} = \int_{\partial\mathcal{O}_2} H_{\partial\mathcal{O}_2} \text{ and} \\ \int_{\partial\mathcal{O}_1} \left( 13H_{\partial\mathcal{O}_1}^2 + 2 \sum_1^{d-1} \kappa_{j,\partial\mathcal{O}_1}^2 \right) &= \int_{\partial\mathcal{O}_2} \left( 13H_{\partial\mathcal{O}_2}^2 + 2 \sum_1^{d-1} \kappa_{j,\partial\mathcal{O}_2}^2 \right). \end{aligned}$$

By the Cauchy-Schwarz inequality, for  $i = 1, 2$ ,

$$\frac{\left( \int_{\partial\mathcal{O}_i} H_{\partial\mathcal{O}_i} \right)^2}{\text{Vol}(\partial\mathcal{O}_i)} \leq \int_{\partial\mathcal{O}_i} H_{\partial\mathcal{O}_i}^2$$

with equality if and only if  $H_{\partial\mathcal{O}_i}$  is constant. Likewise,  $\sum_1^{d-1} \kappa_{j,\partial\mathcal{O}_i}^2 \geq \frac{1}{d-1} \left( \sum_1^{d-1} \kappa_{j,\partial\mathcal{O}_i} \right)^2$ , with equality if and only if  $\kappa_{1,\partial\mathcal{O}_i} = \kappa_{2,\partial\mathcal{O}_i} = \dots = \kappa_{d-1,\partial\mathcal{O}_i}$ . Since equality holds for  $\mathcal{O}_1$ , the three

equalities of (7) mean that  $H_{\partial\mathcal{O}_2}$  is a constant and the principal curvatures of  $\partial\mathcal{O}_2$  are all equal. Thus  $\mathcal{O}_2$  must be the disjointn union of  $l$  balls of fixed radius  $\rho'$ . Again using the first two equalities of (7), we see that we must have  $l = m$  and  $\rho' = \rho$ .  $\square$

We remark that the proof of this theorem is a bit delicate, in the sense that it is important for us that in the expression for  $a_3$  the coefficients of  $H_{\partial\mathcal{O}}^2$  and  $\sum_1^{d-1} \kappa_{j,\partial\mathcal{O}}^2$  have the same sign. However, if  $\mathcal{R}_{\mathcal{O}} = \mathcal{R}_{B(\rho)}$  and we know a priori that  $\mathcal{O}$  is a ( $d$ -dimensional) *convex* set, then we do not need to use the coefficient  $a_3$  to prove that  $\mathcal{O}$  is, up to translation,  $B(\rho)$ . Instead, one can use first two equalities of (7) and the fact that in the Alexandrov-Fenchel inequality

$$\left( \frac{\text{Vol}(\partial\mathcal{O})}{\text{Vol}(\partial B(\rho))} \right)^{1/(d-1)} \leq \left( \frac{\int_{\partial\mathcal{O}} H_{\mathcal{O}}}{\int_{\partial B(\rho)} H_{B(\rho)}} \right)^{1/(d-2)}$$

equality holds if and only if  $\mathcal{O}$  is a ball [2, Chapter 4, Section 9] or [1].

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